NORTH-HOLLAND

Bateman's Equation and Similar Units

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ABSTRACT

The multiplication $K(x, y) \circ F(y, z) = \int K(x, y) F(y, z)\, dy$ of real functions K and F can be interpreted as the analytic version of matrix multiplication. This suggests examining whether this multiplication has a unit element, i.e., a kernel $E(x, y)$ such that $E(x, y) \circ F(y, z) = F(x, z)$ or $\int E(x, y) f(y)\, dy = f(x)$ for infinitely many linear independent functions f. Bateman's function $[\sin(x - y)]/\pi(x - y)$ is an example of such a kernel $E(x, y)$. This paper develops a procedure to construct Bateman's function and similar units. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

The operator

$$f(x) \to \int K(x, y) f(y)\, dy$$

can be interpreted as a multiplication on spaces of functions in two real variables, i.e.,

$$K(x, y) \circ F(y, z) := \int_I K(x, y) F(y, z)\, dy, \qquad I \subset \mathbb{R} \qquad (1.1)$$

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and hence appears as the analytic version of matrix multiplication. This analogy suggests examining whether this multiplication has a unit element, i.e., a kernel $E(x, y)$ such that

$$E(x, y) \circ F(y, z) = F(x, z) \tag{1.2}$$

or

$$\int_I E(x, y) f(y)\, dy = f(x) \tag{1.3}$$

for infinitely many linear independent functions f.

Bateman's equation [1, p. 483, formula (38)]

$$\int_{-\infty}^{\infty} \frac{\sin(x-y)}{\pi(x-y)} f(y)\, dy = f(x) \tag{1.4}$$

provides an example of such a kernel $E(x, y)$. G. H. Hardy commented on this equation in his introduction of [2] as follows:

> In one of his papers on integral equations, Mr. H. Bateman has stated and made use of the equation $f(x) = (1/\pi)\int_{-\infty}^{\infty}(\sin(t-x)/(t-x))f(t)\, dt$, a formula which is striking in itself and capable of interesting applications ... .

The main subject of [2] is to determine extensive classes of "1-functions," i.e. functions f which satisfy Equation (1.4).

In this paper we construct unit elements $E(x, y)$ by the following procedure: We start with the initial part [viewed as an $\mathbb{R}(x)^N$ column vector]

$$F_N(x) := (f_n(x))_{1 \leqslant n \leqslant N}$$

of an appropriate sequence of real functions

$$F(x) := (f_n(x))_{n \geqslant 1}$$

and consider the $\mathbb{R}$-algebra A_N generated by the functions $f_m(x) f_n(y)$, $1 \leqslant m, n \leqslant N$, with the multiplication as defined in Equation (1.1) for $I = I_N$. The symmetric matrix

$$M_N := (\mu_{m,n}^{(N)})_{1 \leqslant m, n \leqslant N} := \left(\int_{I_N} f_m(x) f_n(x)\, dx \right) \tag{1.5}$$

is assumed to be positive definite. Then the unit element of A_N is

$$E_N(x, y) = F_N(x)^t E_N F_N(y) = \sum_{1 \leqslant m, n \leqslant N} \varepsilon_{m,n}^{(N)} f_m(x) f_n(y), \quad (1.6)$$

where

$$E_N := \left(\varepsilon_{m,n}^{(N)}\right)_{1 \leqslant m, n \leqslant N}$$

denotes the inverse of M_N.

The intervals

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

are chosen such that all limits

$$\varepsilon_{m,n} := \lim_{N \to \infty} \varepsilon_{m,n}^{(N)}$$

and hence

$$E := \left(\varepsilon_{m,n}\right)_{m, n \geqslant 1}$$

exist. Then

$$E(x, y) := F(x)^t E F(y) = \sum_{m, n \geqslant 1} \varepsilon_{m,n} f_m(x) f_n(y)$$

is a candidate for a kernel in Equation (1.3) with

$$I := \bigcup_{N \geqslant 1} I_N.$$

The uniquely determined upper Cholesky factorization

$$E_N = H_N^t H_N$$

of E_N as a product of an upper triangular matrix

$$H_N = \left(\eta_{m,n}^{(N)}\right)_{1 \leqslant m, n \leqslant N},$$

with positive diagonal elements, and its transpose H_N^t, yields via

$$(\psi_m^{(N)})_{1 \leqslant m \leqslant N} := H_N F_N(x) = \Big(\sum_{m \leqslant n \leqslant N} \eta_{m,n}^{(N)} f_n(x) \Big)_{1 \leqslant m \leqslant N}$$

an appropriate orthonormal system of functions for an orthonormal representation of $E_N(x, y)$ defined in (1.6), i.e.,

$$E_N(x, y) = \sum_{1 \leqslant m \leqslant N} \psi_m^{(N)}(x) \psi_m^{(N)}(y). \tag{1.7}$$

The system is appropriate in the sense that all limits

$$\eta_{m,n} := \lim_{N \to \infty} \eta_{m,n}^{(N)}$$

and hence

$$H := (\eta_{m,n})_{m,n \geqslant 1}$$

exist. Therefore the orthonormal functions $\psi_m^{(N)}$ can also be carried over to the limit case $N = \infty$:

$$(\psi_m(x))_{m \geqslant 1} := HF(x) = \Big(\sum_{n \geqslant m} \eta_{m,n} f_n(x) \Big)_{m \geqslant 1}.$$

This suggests the representation

$$E(x, y) = \sum_{m \geqslant 1} \psi_m(x) \psi_m(y) \tag{1.8}$$

as the limit case of Equation (1.7).

Starting with $f_n(x) = x^{n-1}$ and $I_N = [-N, N]$ yields

(1) Bateman's kernel

$$E(x, y) = \frac{\sin(x - y)}{\pi(x - y)},$$

(2) the orthonormal system

$$\psi_m(x) = \sqrt{m - \tfrac{1}{2}}\,\frac{J_{m-1/2}(x)}{\sqrt{x}}, \qquad m \geqslant 1,$$

where $J_\nu(x)$ denotes the Bessel function of order ν, and

(3) the addition theorem for the spherical functions

$$\frac{\sin(x-y)}{\pi(x-y)} = \sum_{m \geqslant 1} \left(m - \tfrac{1}{2}\right) \frac{J_{m-1/2}(x)}{\sqrt{x}}\,\frac{J_{m-1/2}(y)}{\sqrt{y}},$$

which is a special case of a formula of Clebsch [6, p. 363, formula (3)] and by which Bateman was originally led to his equation (see [2, p. 447]).

In Section 2 the general procedure in the finite-dimensional case is developed: the trick is to orthonormalize the system $(f_1, \ldots, f_N)$ backwards. The crucial point in the explicit construction of $E_N(x, y)$ is obviously the explicit inversion of the matrix M_N. This is done in Section 3 for two types of matrices on the basis of a theorem of Cauchy. Section 4 contains examples in the limit case $N = \infty$.

We refrain from a discussion of which functions F or f actually satisfy Equation (1.2) or (1.3), and will be content to repeat some known results concerning Bateman's original equation (1.4).

Further notation: For matrices $M = (\nu_{m,n})_{1 \leqslant m, n \leqslant N}$,

$$M(j,k) := \begin{cases} (\nu_{m,n})_{j \leqslant m, n \leqslant k} & \text{if } 1 \leqslant j \leqslant k \leqslant N, \\ 1 & \text{otherwise}, \end{cases}$$

$$M(j,k;r,s) := \begin{cases} (\nu_{m,n})_{\substack{j \leqslant m \leqslant k,\, m \neq r \\ j \leqslant n \leqslant k,\, n \neq s}} & \text{if } j \leqslant r,\ s \leqslant k, \text{ and } j < k, \\ 1 & \text{if } j = r = s = k, \\ 0 & \text{otherwise}. \end{cases}$$

$$L_N := \left(\frac{(-1)^{m+n} |M(1,m;n,m)|}{\sqrt{|M(1,m-1)|\,|M(1,m)|}} \right)_{1 \leqslant m, n \leqslant N} \tag{1.9}$$

with $M = M_N$ a lower triangular matrix with positive diagonal elements, and

$$H_N := \left(\eta_{m,n}^{(N)}\right) := \left(\frac{(-1)^{m+n} |M(m, N; n, m)|}{\sqrt{|M(m, N)| |M(m+1, N)|}} \right)_{1 \leqslant m, n \leqslant N}$$

with $M = M_N$ an upper triangular matrix with positive diagonal elements. $\delta_{m,n}$ denotes the Kronecker symbol, and

$$\delta_{m \equiv n} := \begin{cases} 1 & \text{if} \quad m \equiv n \pmod 2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta_{m \leqslant n} := \begin{cases} 1 & \text{if} \quad m \leqslant n, \\ 0 & \text{otherwise}. \end{cases}$$

2. FORWARD AND BACKWARD ORTHONORMALIZATION

PROPOSITION 1.

(1) *Forward orthonormalization of* $(f_1, \dots, f_N)$: *The functions* $\phi_m^{(N)} : I_N \to \mathbb{R}$ *defined by*

$$\left(\phi_m^{(N)}(x)\right)_{1 \leqslant m \leqslant N} := L_N F_N(x)$$

are orthonormal: $\int_{I_N} \phi_m^{(N)}(x) \phi_n^{(N)}(x)\, dx = \delta_{m,n}$.

(2) *Backward orthonormalization of* $(f_1, \dots, f_N)$: *The functions* $\psi_m^{(N)} : I_N \to \mathbb{R}$ *defined by*

$$\left(\psi_m^{(N)}(x)\right)_{1 \leqslant m \leqslant N} := H_N F_N(x)$$

are orthonormal: $\int_{I_N} \psi_m^{(N)}(x) \psi_n^{(N)}(x)\, dx = \delta_{m,n}$.

Proof. (1): By the definition of L_N in Equation (1.9),

$$\phi_m^{(N)}(x) = \sum_{1 \leqslant j \leqslant m} (-1)^{m+j} \frac{|M(1, m; j, m)|}{\sqrt{|M(1, m-1)| |M(1, m)|}} f_j(x).$$

Hence for $1 \leqslant m \leqslant n \leqslant N$

$$
\begin{aligned}
&\sqrt{|M(1, m-1)||M(1, m)|}\sqrt{|M(1, n-1)||M(1, n)|}\\
&\quad\times \int_{I_N} \phi_m^{(N)}(x)\phi_n^{(N)}(x)\,dx\\
&\quad= \int_{I_N}\Bigg(\sum_{1\leqslant i\leqslant m}(-1)^{m+i}|M(1, m; i, m)|f_i(x)\Bigg)\\
&\qquad\times\Bigg(\sum_{1\leqslant j\leqslant n}(-1)^{n+j}|M(1, n; j, n)|f_j(x)\Bigg)\,dx\\
&\quad= \sum_{1\leqslant i\leqslant m}(-1)^{m+i}|M(1, m; i, m)|\\
&\qquad\times\Bigg(\sum_{1\leqslant j\leqslant n}(-1)^{n+j}\mu_{ij}^{(N)}|M(1, n; j, n)|\Bigg) \qquad \text{by } (1.5)\\
&\quad= \sum_{1\leqslant i\leqslant m}(-1)^{m+i}|M(1, m; i, m)|\cdot\begin{vmatrix}\mu_{11}^{(N)} & \cdots & \mu_{1,n-1}^{(N)} & \mu_{1i}^{(N)}\\ \vdots & & \vdots & \vdots\\ \mu_{n1}^{(N)} & \cdots & \mu_{n,n-1}^{(N)} & \mu_{ni}^{(N)}\end{vmatrix}
\end{aligned}
$$

and, since the last column of the μ-determinant is duplicated for each i, $1 \leqslant i \leqslant m \leqslant n-1$,

$$
= \begin{cases} 0 & \text{if} \quad m < n,\\ |M(1, n-1)||M(1, n)| & \text{if} \quad m = n. \end{cases}
$$

(2): Orthonormalize $(f_N, \ldots, f_1)$ forward according to Proposition 1(1), and denote the resulting system by $(\psi_N^{(N)}, \ldots, \psi_1^{(N)})$. ■

REMARKS.

(1) $(\psi_1^{(N)}, \ldots, \psi_N^{(N)})$ is a maximal orthonormal system in A_N; hence by

Proposition 1(2)

$$E_N(x, y) = \sum_{1 \leqslant m \leqslant N} \psi_m^{(N)}(x)\psi_m^{(N)}(y) = F_N(x)^t H_N^t H_N F_N(y).$$

A comparison with the representation of $E_N(x, y)$ in Equation (1.6) shows

$$E_N = H_N^t H_N,$$

which is the upper Cholesky factorization of E_N. Similarly

$$E_N(x, y) = \sum_{1 \leqslant m \leqslant N} \phi_m^{(N)}(x)\phi_m^{(N)}(y) = F_N(x)^t L_N^t L_N F_N(y)$$

according to Proposition 1(1), which gives the lower Cholesky factorization

$$E_N = L_N^t L_N.$$

(2) Orthonormalization of the functions

$$G_N(x) := \left(g_m^{(N)}(x)\right)_{1 \leqslant m \leqslant N} := E_N F_N(x) \tag{2.1}$$

forwards according to Proposition 1(1) yields the same orthonormal system $(\psi_1^{(N)}, \ldots, \psi_N^{(N)})$ as orthonormalization of $(f_1, \ldots, f_N)$ backwards according to Proposition 1(2). That is,

$$L'_N G_N(x) = \left(\psi_m^{(N)}(x)\right)_{1 \leqslant m \leqslant N}, \tag{2.2}$$

where L'_N denotes the matrix L_N in Equation (1.9) with $M = E_N$ instead of $M = M_N$, for here we have

$$\int_{I_N} g_m^{(N)}(x) g_n^{(N)}(x)\, dx = \varepsilon_{m,n}^{(N)}.$$

To verify Equation (2.2) it is sufficient to observe that by Equation (2.1) $L'_N G_N(x) = L'_N E_N F_N(x)$ and that—like H_N—$L'_N E_N$ is an upper triangular matrix with positive diagonal elements. Both matrices $L'_N E_N$ and H_N transform $(f_1, \ldots, f_N)$ into an orthonormal system; hence they must be equal.

The representation (2.2) of the functions $\psi_m^{(N)}$ in terms of $(g_1^{(N)}, \dots, g_N^{(N)})$ with the coefficients of L'_N is sometimes more favorable than that of Proposition 1(2) in terms of $(f_1, \dots, f_N)$ with the coefficients of H_N—see the example $f_n(x) = e^{-n^2 x}$ in Section 4.

3. TWO CONCRETE CASES

We first establish some necessary technical results.

PROPOSITION 2.

(1) $$\det\left(\frac{1}{a_m + b_n}\right)_{1 \leqslant m, n \leqslant N} = \frac{\prod_{1 \leqslant m < n \leqslant N}(a_m - a_n)(b_m - b_n)}{\prod_{1 \leqslant m, n \leqslant N}(a_m + b_n)}.$$

(2) *The inverse of the matrix* $M = \left(\dfrac{1}{a_m + a_n}\right)_{1 \leqslant m, n \leqslant N}$ *is*

$$M^{-1} = \left(\frac{A_m A_n}{a_m + a_n}\right)$$

with $A_m = \prod_{1 \leqslant j \leqslant N}(a_m + a_j) \Big/ \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq m}}(a_m - a_j)$.

(3) *The inverse of the matrix* $M = \left(\dfrac{\delta_{m \equiv n}}{a_m + a_n}\right)_{1 \leqslant m, n \leqslant N}$ *is*

$$M^{-1} = \left(\delta_{m \equiv n} \frac{A_m A_n}{a_m + a_n}\right)$$

with $A_m = \prod_{\substack{1 \leqslant j \leqslant N \\ j \equiv m\,(2)}}(a_m + a_j) \Big/ \prod_{\substack{1 \leqslant j \leqslant N \\ j \equiv m\,(2),\, j \neq m}}(a_m - a_j)$.

Proof. (1) is a theorem of Cauchy [3, p. 87]. (2) and (3) are immediate consequences of (1). ■

REMARK. Suppose that $0 < a_1 < a_2 < \cdots$. Then, referring to Proposition 2 above,

$$A_m A_n = (-1)^{m-n} |A_m A_n|$$

in part (2), and

$$A_m A_n = (-1)^{(m-n)/2} |A_m A_n| \qquad \text{if} \quad m \equiv n \pmod 2)$$

in part (3).

CASE 1. The functions $(f_n(x))_{n \geqslant 1}$ and the intervals I_N are chosen such that for all N

$$M_N = \left(\frac{c_m(N) c_n(N)}{a_m + a_n} \right)_{1 \leqslant m, n \leqslant N}$$

with $0 < a_1 < a_2 < \cdots$ and $c_m(N) > 0$.

With the notation

$$A(n, m) := \prod_{1 \leqslant j \leqslant n} (a_m + a_j) \Bigg/ \prod_{\substack{1 \leqslant j \leqslant n \\ j \neq \min(m, n)}} |a_m - a_j|, \tag{3.1}$$

$$\alpha_m^{(N)} := \frac{A(N, m)}{c_m(N)}, \quad 1 \leqslant m \leqslant N, \quad \text{and} \quad \alpha_m := \lim_{N \to \infty} \alpha_m^{(N)}, \tag{3.2}$$

Proposition 2 shows

$$E_N = \left((-1)^{m+n} \frac{\alpha_m^{(N)} \alpha_n^{(N)}}{a_m + a_n} \right)_{1 \leqslant m, n \leqslant N}, \tag{3.3}$$

$$H_N = \left(\delta_{m \leqslant n} (-1)^{m+n} \frac{\sqrt{2a_m}}{A(m,n)} \alpha_n^{(N)} \right)_{1 \leqslant m, n \leqslant N}, \tag{3.4}$$

$$L'_N = \left(\delta_{n \leqslant m} \frac{\sqrt{2a_m}\, A(m,n)}{\alpha_n^{(N)} (a_m + a_n)} \right)_{1 \leqslant m, n \leqslant N}. \tag{3.5}$$

E_N contains the coefficients $\varepsilon_{m,n}^{(N)}$ in the representation of $E_N(x, y)$ according to Equation (1.6). H_N $[L'_N]$ contains the coefficients of the functions $\psi_m^{(N)}$ in terms of $f_1, \dots, f_N$ $[g_1^{(N)}, \dots, g_N^{(N)}]$ according to Proposition 1(2) [Equation (2.2)]. Finally, the functions $g_m^{(N)}$ are normalized to

$$h_m^{(N)}(x) := \frac{(-1)^m}{\alpha_m^{(N)}} g_m^{(N)}(x), \qquad 1 \leqslant m \leqslant N,$$

i.e. such that

$$\int_{I_N} h_m^{(N)}(x) h_n^{(N)}(x)\, dx = \frac{1}{a_m + a_n}. \tag{3.6}$$

Then the equations (3.3), (3.4), (3.5), (2.1), and (2.2) imply:

PROPOSITION 3. *Under Case* 1,

$$\begin{aligned} E_N(x, y) &= \sum_{1 \leqslant m, n \leqslant N} (-1)^{m+n} \frac{\alpha_m^{(N)} \alpha_n^{(N)}}{a_m + a_n} f_m(x) f_n(y) \\ &= \sum_{1 \leqslant m \leqslant N} \psi_m^{(N)}(x) \psi_m^{(N)}(y) \end{aligned}$$

with the orthonormal functions

$$\begin{aligned} \psi_m^{(N)}(x) &= \sqrt{2a_m} \sum_{m \leqslant n \leqslant N} (-1)^{n-m} \frac{\alpha_n^{(N)}}{A(m,n)} f_n(x) \\ &= \sqrt{2a_m} \sum_{1 \leqslant n \leqslant m} (-1)^n \frac{A(m,n)}{a_m + a_n} h_n^{(N)}(x), \end{aligned}$$

and

$$h_n^{(N)}(x) = \sum_{1 \leqslant m \leqslant N} (-1)^m \frac{\alpha_m^{(N)}}{a_m + a_n} f_m(x).$$

The corresponding limit-case formulae are

$$E(x, y) = \sum_{m, n \geqslant 1} (-1)^{m+n} \frac{\alpha_m \alpha_n}{a_m + a_n} f_m(x) f_n(y),$$

$$\psi_m(x) = \sqrt{2a_m} \sum_{n \geqslant m} (-1)^{n-m} \frac{\alpha_n}{A(m, n)} f_n(x)$$

$$= \sqrt{2a_m} \sum_{1 \leqslant n \leqslant m} (-1)^n \frac{A(m, n)}{a_m + a_n} h_n(x),$$

and

$$h_n(x) = \sum_{m \geqslant 1} (-1)^m \frac{\alpha_m}{a_m + a_n} f_m(x).$$

CASE 2. The functions $(f_n(x))_{n \geqslant 1}$ and the intervals I_N are chosen such that for all N

$$M_N = \left(\delta_{m \equiv n} \frac{c_m(N) c_n(N)}{a_m + a_n} \right)_{1 \leqslant m, n \leqslant N}$$

with $0 < a_1 < a_2 < \cdots$ and $c_m(N) > 0$.

Then with the modified notation

$$A(n, m) := \prod_{\substack{1 \leqslant j \leqslant n \\ j \equiv m\,(2)}} (a_m + a_j) \Bigg/ \prod_{\substack{1 \leqslant j \leqslant n \\ j \equiv m\,(2) \\ j \neq \min(m, n)}} |a_m - a_j|, \qquad (3.1')$$

and again with

$$\alpha_m^{(N)} := \frac{A(N, m)}{c_m(N)}, \quad 1 \leqslant m \leqslant N, \quad \text{and} \quad \alpha_m := \lim_{N \to \infty} \alpha_m^{(N)}, \quad (3.2')$$

Proposition 2 shows

$$E_N = \left(\delta_{m \equiv n} (-1)^{(n-m)/2} \frac{\alpha_m^{(N)} \alpha_n^{(N)}}{a_m + a_n} \right)_{1 \leqslant m, n \leqslant N}, \tag{3.3$'$}$$

$$H_N = \left(\delta_{m \leqslant n} \delta_{m \equiv n} (-1)^{(n-m)/2} \frac{\sqrt{2a_m}}{A(m,n)} \alpha_n^{(N)} \right)_{1 \leqslant m, n \leqslant N}, \tag{3.4$'$}$$

and hence

PROPOSITION 3$'$. *Under Case* 2,

$$E_N(x, y) = \sum_{\substack{1 \leqslant m, n \leqslant N \\ m \equiv n\,(2)}} (-1)^{(n-m)/2} \frac{\alpha_m^{(N)} \alpha_n^{(N)}}{a_m + a_n} f_m(x) f_n(y)$$

$$= \sum_{1 \leqslant m \leqslant N} \psi_m^{(N)}(x) \psi_m^{(N)}(y)$$

with the orthonormal functions

$$\psi_m^{(N)}(x) = \sqrt{2a_m} \sum_{\substack{m \leqslant n \leqslant N \\ n \equiv m\,(2)}} (-1)^{(n-m)/2} \frac{\alpha_n^{(N)}}{A(m,n)} f_n(x).$$

The corresponding limit-case formulae are

$$E(x, y) = \sum_{\substack{m, n \geqslant 1 \\ m \equiv n\,(2)}} (-1)^{(n-m)/2} \frac{\alpha_m \alpha_n}{a_m + a_n} f_m(x) f_n(y),$$

$$\psi_m(x) = \sqrt{2a_m} \sum_{\substack{n \geqslant m \\ n \equiv m\,(2)}} (-1)^{(n-m)/2} \frac{\alpha_n}{A(m,n)} f_n(x).$$

4. EXAMPLES

EXAMPLE 1. $f_n(x) := x^{n-1}$, $I_N := [-N/c, N/c]$, $c > 0$. Hence

$$M_N = \left(\delta_{m \equiv n} \frac{2(N/c)^{m+n-1}}{m+n-1} \right)_{1 \leqslant m,\, n \leqslant N}.$$

This corresponds to Case 2 with

$$a_m = m - \tfrac{1}{2} \quad \text{and} \quad c_m(N) = \sqrt{2}\,(N/c)^{m-1/2}.$$

The equations (3.1′) and (3.2′) show that

$$\alpha_m^{(N)} = \frac{1}{\sqrt{2}} \left(\frac{c}{N} \right)^{m-1/2} \left(\prod_{\substack{1 \leqslant j \leqslant N \\ j \equiv m\,(2)}} (m+j-1) \Bigg/ \prod_{\substack{1 \leqslant j \leqslant N \\ j \equiv m\,(2) \\ j \neq m}} |m-j| \right),$$

and the Wallis product formula yields

$$\alpha_m = \sqrt{\frac{c}{\pi}}\, \frac{c^{m-1}}{(m-1)!}.$$

Now Proposition 3′ gives

$$E(x,y) = \frac{1}{\pi} \sum_{\substack{m,\, n \geqslant 0 \\ m \equiv n\,(2)}} (-1)^{(n-m)/2} \frac{c^{m+n+1}}{(m+n+1)m!n!} x^m y^n$$

$$= \frac{1}{\pi} \sum_{r \geqslant 0} \sum_{0 \leqslant n \leqslant 2r} \frac{(-1)^{r-n} c^{2r+1}}{(2r+1)(2r-n)!n!} x^{2r-n} y^n$$

$$= \frac{1}{\pi} \sum_{r \geqslant 0} \frac{(-1)^r c^{2r+1}}{(2r+1)!} (x-y)^{2r};$$

hence

$$E(x, y) = \frac{\sin c(x - y)}{\pi(x - y)}$$

and

$$\psi_m(x) = \sqrt{\frac{(2m - 1)c}{\pi}}$$
$$\times \sum_{\substack{n \geqslant m \\ n \equiv m\,(2)}} (-1)^{(n-m)/2} \frac{2^m}{(n + m)!} \frac{[(n + m)/2]!}{[(n - m)/2]!} (cx)^{n-1}.$$

Using Legendre's duplication formula, it follows that

$$\psi_m(x) = \sqrt{m - \tfrac{1}{2}} \frac{J_{m-1/2}(cx)}{\sqrt{x}}, \qquad m \geqslant 1.$$

In particular, the limit case of the orthonormal representation (1.7), i.e. Equation (1.8), becomes

$$\frac{\sin c(x - y)}{\pi(x - y)} = \frac{1}{\sqrt{xy}} \sum_{m \geqslant 1} (m - \tfrac{1}{2}) J_{m-1/2}(cx) J_{m-1/2}(cy).$$

Concerning the orthonormality of the functions ψ_m, $m \geqslant 1$, see [2, p. 448].

Some examples of functions f which satisfy Bateman's equation (1.4):

$$f(x) = 1,$$

$$f(x) = \cos ax, \; f(x) = \sin ax, \quad |a| < 1 \qquad [5, \text{p. } 350],$$

$$f(x) = \frac{\sin ax}{x}, \quad |a| \leqslant 1 \qquad [2, \text{p. } 447],$$

$$f(x) = x^{-m} J_n(x), \quad 0 \leqslant m \leqslant n \qquad [5, \text{p. } 352],$$

$$f(x) = x^{-\beta} J_\beta(x), \quad \beta > -\tfrac{1}{2} \qquad [2, \text{p. } 460].$$

REMARKS.

(1) $E(x, y) \circ E(y, z) = E(x, z)$. (See [2, p. 449, formula (4)]. Also see Hardy's footnote on p. 451 and Section VI of [2].)

(2) A necessary and sufficient condition for a function $f(x)$ of $L^2(-\infty, \infty)$ to be a solution of Equation (1.4) is that it should be of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} F(t) e^{-ixt}\, dt, \text{ where } F \text{ is } L^2(-1, 1)$$

[5, p. 350, Theorem 156].

(3) In particular,

$$\frac{1}{\sqrt{x}} J_{n+1/2}(x) = \frac{i^n}{\sqrt{2\pi}} \int_{-1}^{1} P_n(t) e^{-ixt}\, dt,$$

where $P_n(t)$ are the Legendre polynomials [4, p. 231]. Thus the functions ψ_m, $m \geqslant 1$, with $c = 1$ satisfy Bateman's equation.

EXAMPLE 2. $f_n(x) := x^{rn+s}$, $r, s \in \mathbb{Z}$, $r \geqslant 1$, $s \geqslant -r$, r odd,

$$I_N := \left[-(N/c)^{1/r}, (N/c)^{1/r}\right], \qquad c > 0.$$

Set

$$\rho := \frac{2s + 1}{2r}.$$

Then

$$M_N = \left(\delta_{m \equiv n} \frac{2(N/c)^{m+n+2\rho}}{r(m + n + 2\rho)} \right)_{1 \leqslant m, n \leqslant N};$$

hence this corresponds to Case 2 with

$$a_m = r(m + \rho) \quad \text{and} \quad c_m(N) = \sqrt{2}\,(N/c)^{m+\rho}.$$

The Equations (3.1′) and (3.2′) show

$$\alpha_m = \frac{\sqrt{2}\,r}{\Gamma([(m+1)/2])\Gamma([(m+2)/2]+\rho)}\left(\frac{c}{2}\right)^{m+\rho},$$

where Γ denotes the classical gamma function. Therefore Proposition 3′ yields in the limit case

$$\psi_m(x) = \sqrt{r(m+\rho)}\,\frac{J_{m+\rho}(cx^r)}{\sqrt{x}}, \qquad m \geqslant 1,$$

and

$$E(x, y) = \frac{r}{\sqrt{xy}} \sum_{m \geqslant 1} (m+\rho) J_{m+\rho}(cx^r) J_{m+\rho}(cy^r).$$

A closed representation for $E(x, y)$ is (for $x \neq y$)

$$E(x, y) = \frac{rc}{2}\frac{(xy)^{r-1/2}}{x^r - y^r}\left[J_{1+\rho}(cx^r)J_\rho(cy^r) - J_\rho(cx^r)J_{1+\rho}(cy^r)\right],$$

which can be verified directly by comparing the coefficients of the identity

$$\begin{aligned}(X-Y)&\sum_{m\geqslant 1}(m+\rho)J_{m+\rho}(2X)J_{m+\rho}(2Y)\\ &= XY\left[J_{1+\rho}(2X)J_\rho(2Y) - J_\rho(2X)J_{1+\rho}(2Y)\right].\end{aligned}$$

REMARK. The example $f_n(x) := x^{2rn+s}$, $r > 0$, $s \geqslant -2r$,

$$I_N := \left[0, (2N/c)^{1/r}\right], \qquad c > 0,$$

corresponds to Case 1 and may be handled using a similar argument. The result is

$$E(x, y) = \frac{2r}{\sqrt{xy}} \sum_{m \geqslant 1} (2m+\rho) J_{2m+\rho}(cx^r) J_{2m+\rho}(cy^r),$$

and, in closed representation,

$$E(x, y) = rc\frac{(xy)^{r-1/2}}{x^{2r} - y^{2r}}\left[y^r J_{1+\rho}(cx^r)J_\rho(cy^r) - x^r J_\rho(cx^r)J_{1+\rho}(cy^r)\right].$$

EXAMPLE 3. $f_n(x) := e^{-n^2 x}$, $I_N = I := [0, \infty)$. This gives

$$M_N = \left(\frac{1}{m^2 + n^2}\right)_{1 \leqslant m, n \leqslant N},$$

which corresponds to Case 1 with

$$a_m = m^2 \quad \text{and} \quad c_m(N) = 1.$$

Hence by Equations (3.1) and (3.2)

$$\alpha_m^{(N)} = \frac{2(N!)^2 m^2}{(N-m)!(N+m)!} \prod_{1 \leqslant j \leqslant N}\left(1 + \frac{m^2}{j^2}\right),$$

and Euler's sine product formula shows

$$\alpha_m = \frac{m}{\pi}\left(e^{\pi m} - e^{-\pi m}\right).$$

Thus Proposition 3 yields

$$E(x, y) = \frac{1}{\pi^2} \sum_{m, n \in \mathbb{Z}} (-1)^{m+n} \frac{mn}{m^2 + n^2} e^{\pi m - m^2 x + \pi n - n^2 y}$$

and

$$\psi_m(x) = \sqrt{2}\,\frac{m}{\pi} \sum_{\substack{n \in \mathbb{Z} \\ |n| \geqslant m}} (-1)^{n-m} n \frac{\prod_{1 \leqslant j < m}(n^2 - j^2)}{\prod_{1 \leqslant j \leqslant m}(n^2 + j^2)} e^{\pi n - n^2 x}$$

$$= \sqrt{2}\, m \sum_{1 \leqslant n \leqslant m} (-1)^n \frac{\prod_{1 \leqslant j < m}(n^2 + j^2)}{\prod_{1 \leqslant j \leqslant m, j \neq n}(n^2 - j^2)} h_n(x) \qquad (4.1)$$

with

$$h_n(x) = \frac{1}{\pi} \sum_{m \in \mathbb{Z}} (-1)^m \frac{m}{m^2 + n^2} e^{\pi m - m^2 x}, \qquad n \geqslant 1.$$

In addition set

$$h_0(x) := \frac{1}{\pi} \sum_{m \in \mathbb{Z},\, m \neq 0} (-1)^m \frac{1}{m} e^{\pi m - m^2 x}.$$

Then, for $x > 0$ and $y > 0$, it is a straightforward check that

$$E(x, y) = \int_0^\infty h_0'(x + z) h_0'(y + z)\, dz.$$

Hence, with

$$H(x, y) := -h_0'(x + y),$$

we have the factorization

$$E(x, z) = H(x, y) \circ H(y, z).$$

The coefficients of M_N here show that Equation (3.6) becomes

$$\int_0^\infty h_m^{(N)}(x) h_n^{(N)}(x)\, dx = \frac{1}{m^2 + n^2}, \qquad 1 \leqslant m, n \leqslant N. \tag{4.2}$$

The limit case of this formula is

$$\int_0^\infty h_m(x) h_n(x)\, dx = \frac{1}{m^2 + n^2}, \qquad m, n \geqslant 1. \tag{4.3}$$

Equation (4.3) can be verified by proving for $c > 0$

$$(m^2 + n^2) \int_c^\infty h_m(x) h_n(x)\, dx$$
$$= -h_m(c) h_n(c) - \int_c^\infty h_0'(x) h_m(x)\, dx - \int_c^\infty h_0'(x) h_n(x)\, dx$$

and further for $n \geqslant 1$,

$$\lim_{c \downarrow 0} h_n(c) = 0 \quad \text{and} \quad \lim_{c \downarrow 0} \int_c^\infty h_0'(x) h_n(x)\, dx = -\frac{1}{2},$$

which are applications of the theta-function transformation formula

$$\sum_{n \in \mathbb{Z}} e^{n^2 \pi i \tau + 2niz} = \frac{1}{\sqrt{-i\tau}} \sum_{n \in \mathbb{Z}} e^{(z - n\pi)^2/(\pi i \tau)}$$

[7, p. 476].

Now the finite-dimensional theory can be used to prove

$$\int_0^\infty \psi_m(x)\psi_n(x)\,dx = \delta_{m,n}. \tag{4.4}$$

For Equation (4.1) and Proposition 3 assert that the representation of ψ_m in terms of $h_1, \dots, h_m$ has the same coefficients as $\psi_m^{(N)}$ in terms of $h_1^{(N)}, \dots, h_m^{(N)}$. Hence the Equations (4.2) and (4.3) and the orthonormality of the functions $\psi_m^{(N)}$ imply Equation (4.4).

REMARK. This example can be modified so that the interval I becomes compact:

$$f_n(x) := x^{n^2 - 1/2} \qquad \text{with} \quad I_N = I := [0, 1]$$

yields the same matrix

$$M_N = \left(\frac{1}{m^2 + n^2} \right)$$

and hence generates the unit element

$$E(x, y) = \frac{1}{\pi^2 \sqrt{xy}} \sum_{m, n \in \mathbb{Z}} (-1)^{m+n} \frac{mn}{m^2 + n^2} e^{\pi m + \pi n} x^{m^2} y^{n^2}.$$

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